Regular black holes in quadratic gravity

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The first-order correction of the perturbative solution of the coupled equations of the quadratic gravity and nonlinear electrodynamics is constructed, with the zeroth-order solution coinciding with the ones given by Ayón-Beato and García and by Bronnikov. It is shown that a simple generalization of the Bronnikov's electromagnetic Lagrangian leads to the solution expressible in terms of the polylogarithm functions. The solution is parametrized by two integration constants and depends on two free parameters. By the boundary conditions the integration constants are related to the charge and total mass of the system as seen by a distant observer, whereas the free parameters are adjusted to make the resultant line element regular at the center. It is argued that various curvature invariants are also regular there that strongly suggests the regularity of the spacetime. Despite the complexity of the problem the obtained solution can be studied analytically. The location of the event horizon of the black hole, its asymptotics and temperature are calculated. Special emphasis is put on the extremal configuration.

I. INTRODUCTION

One of the most important and intriguing questions of the black hole physics is the problem of singularities that reside in their internal region, hidden to an external observer by the event horizon. In the vast majority of papers singularities are treated as symptoms of illness of the theory rather than its health (see however Ref. [1] for a different point of view), and, consequently, a great deal of efforts were directed to constructing singularity-free models. However, a subtle point is that the Einstein field equations loose their predicative power and cannot be trusted when the curvature of the manifold approaches the Planck regime. Indeed, according to our present understanding a proper description of the gravitational phenomena should be given by the quantum gravity, being perhaps a part of a more fundamental theory. And although at the present stage we have no clear idea how this theory looks like, we expect that the action functional describing its low-energy approximation should consist of higher order terms constructed from the curvature and its covariant derivatives to some required order. It means that in the full theory the analogs of the nonsingular solution of the Einstein gravity may loose their nonsingular character as well as the singular ones their singularities.

Among various modifications of the general relativity the prominent role is played by the quadratic gravity [2, 3, 4, 5, 6, 7, 8]. Motivations for introducing into the action functional terms which are quadratic in curvature are numerous. For example, when invented, the equations of quadratic gravity have been treated as an exact formulation of the theory of gravitation. For historic informations and important references the reader is referred to Ref. [9]. It may be considered, quite naturally, as truncation of series expansion of the action of the more general theory. Such terms appear generically in one-loop calculations of the quantum field theory in curved background [10]. Moreover, from the point of view of the semi-classical gravity it might be treated (in certain circumstances) as some sort of a poor man's stress- energy tensor, allowing in a relatively simple way to mimic, especially when the application of the full stress- energy tensor would produce extremely complicated or even intractable results, the fairly more complex source term of the field equations.

Analyses of the spherically-symmetric and static solutions to the higher derivative theory has been carried out in [3, 11, 12, 13, 14, 15]. For example, in Ref [3] it has been shown that the weak-field limit of the quadratic gravity involves, beside the ordinary Newtonian term, also the terms with the Yukawa-like potential. Series solutions near

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the $r=0$ have been investigated in [3, 14]. Such solutions are limited to the closest neighborhood of the center and should be matched to the appropriate solutions valid for large r that require numerical integration. However, an important lesson that follows from this calculations is that the regular solutions of the equations of the higher order theories are quite common [3].

As compared with the General Relativity, the Lagrangian of the quadratic gravity in four dimensions requires two additional terms $\alpha R_{ab}R^{ab}+\beta R^2$, where α and β are the coupling constants. Since the coupling constants are expected to be very small, one can easily device a perturbative approach to the problem treating the classical solution of the Einstein field equations as the zeroth-order of the approximation. Successive perturbations are therefore solutions of the differential equations of ascending complexity. It should be noted that although the method is clear the calculations beyond the first- order may be intractable.

According to the well-known theorem, if the Lagrangian $\mathcal{L}(F)$ has Maxwell asymptotics for weak fields, i.e., $\mathcal{L}(F) \sim F$ and $\mathcal{L}_F = d\mathcal{L}/dF \to 1$ as $F \to 0$, then any static and spherically-symmetric solutions to the system of coupled equations of nonlinear electrodynamics and general relativity characterized by electric charge, Q_e, cannot have a regular center [16, 17, 18]. This no-go theorem does not forbid existence of the solutions with magnetic charge, Q, as well as some hybrid solutions in which the electric field does not extend to the central region. On the other hand, however, it has been argued in Ref. [19] that the Maxwell asymptotics at great distances are essentially different from these at the center, and, consequently, the condition $\mathcal{L}_F \to 1$ as $F \to 0$ is too restrictive.

The issue of the regular black holes in general relativity has a long and interesting history. For example, one of the method that can be used in construction of such configurations is simply replacing the singular black hole interior by a regular core. This idea appeared almost forty years ago, in mid sixties [20, 21, 22] and is actively investigated today. In models considered in Refs. [23, 24] part of the region inside the event horizon is joined through a thin boundary layer to de Sitter geometry. Such a geometric surgery certainly does not exhaust all interesting possibilities: of equal importance are the regular geometries with suitably chosen profile functions, or, better, exact solutions constructed for specific, physically reasonable sources [25, 26, 27, 28, 29, 30]. One of the most intriguing regular solutions to the coupled equations of the nonlinear electrodynamics and gravity have been constructed by Ayón-Beato and García [31] and by Bronnikov [17]. We shall refer to solutions of this type as ABGB geometry. The former describes a regular static and spherically symmetric configuration with an electric charge whereas the latter describes a similar geometry characterized by the total mass M and Q. For certain values of the parameters both solutions describe a black hole. It should be noted that the electric solution does not contradict the non existence theorem as the formulation of nonlinear electrodynamics employed by Ayón-Beato and García ($\mathcal{P}$ framework in the nomenclature of Refs. [17, 18]) is not the one to which one refers in the proviso of the no-go theorem. Indeed, the solution of Ayón-Beato and García has been constructed in a formulation of the nonlinear electrodynamics obtained from the original one ($\mathcal{F}$ framework) by means of a Legendre transformation (see Ref. [18] for details). An attractive feature of this solutions that certainly simplifies calculations is possibility to express the location of the horizons in terms of the known special functions [32, 33].

The natural question that arises in connection with the foregoing discussion is whether or not it is possible to construct an analog of the solution of the ABGB-type in the quadratic gravity which shares with its classical counterpart regularity at $r=0$. And although the full, detailed answer is beyond our capabilities, it is possible to provide an affirmative answer to the restricted problem. Indeed, since the complexity of the coupled equations of the quadratic gravity and nonlinear electrodynamics, even in the simplest case of spherically-symmetric and static geometries, hinders construction of the exact solution, one has to refer to the analytical approximations or numerical methods.

In this paper we shall employ perturbative methods to construct the first-order solution to the equations of the quadratic gravity with the source term being generalization of the stress-energy tensor of the Bronnikov type. The zeroth-order solution coincides, as expected, with the ABGB line element whereas the first-order correction can be elegantly expressed in terms of the polylogarithm functions. An interesting feature of this very solution is its regularity for $r=0$. The Kretschmann scalar and other curvature invariants are also regular at the center that suggests regularity of the underlying geometry. It should be emphasized that the demonstration of the regularity of the full solution would require either profound understanding of the perturbation series to any required order or construction of the full, physically acceptable nonperturbative solution. On the other hand, the perturbative approach is expected to yield reasonable results especially for higher derivative dynamical equations. In fact it may be the only method to deal with them. Indeed, since the quadratic gravity involves fourth-order derivatives of the metric their nonperturbative solutions may appear to be spurious and one has to invent a method for systematic selecting physical ones. It seems that the acceptable solutions, when expanded in powers of the small parameter, should reduce to those obtained within the framework of perturbative approach [34, 35, 36].

The paper is organized as follows. In Sec. II we introduce basic equations and briefly sketch employed method. We choose the line element in the form propounded by Visser [37], which has proved to be a very useful representation considerably simplifying calculations. In Sec. III we introduce the Lagrangian of the nonlinear electrodynamics, construct solutions to the first-order equations and establish regularity of the thus obtained line element. Subsequently we study the limit of the vanishing Q and analyse the regularity of various curvature invariants. Corrections to the location of the inner and outer horizons and to temperature are given in Sec. IV A. The position of the horizons of the ABGB spacetime is given in terms of the real branches of the Lambert functions. The extremal configuration is studied in Sec. IV B. Specifically, we calculate modifications of the location of the degenerate horizon caused by quadratic gravity and analyse relations between Q and the total mass as seen by a distant observer. Sec. V contains final remarks. In Appendix we have collected useful formulas and presented a brief description of the method of integration of the field equations in terms of the polylogarithm functions. Throughout the paper the geometric system of units has been adopted and the the sign conventions are taken to be that of MTW [38].

II. THE EQUATIONS

In absence of the cosmological term, the coupled system of the nonlinear electrodynamics and the quadratic gravity is described by the action

$$S = \frac{1}{16\pi G} S_g + S_m, \tag{1}$$

where

$$S_g = \int \left(R + \alpha R^2 + \beta R_{ab} R^{ab}\right) \sqrt{-g}\, d^4x, \tag{2}$$

and

$$S_m = -\frac{1}{16\pi} \int \mathcal{L}\left(F\right) \sqrt{-g}\, d^4x. \tag{3}$$

Here $\mathcal{L}(F)$ is some functional of $F = F_{ab}F^{ab}$ (its exact form will be given later) and all symbols have their usual meaning. The third possible term constructed form the Kretschmann scalar, $\gamma R_{abcd}R^{abcd}$, may by removed from the Lagrangian with the help of the Gauss-Bonnet invariant. Of numerical parameters α and β we assume, as usual, that they are small and of comparable order, otherwise they would lead to the observational consequences within our solar system. Their ultimate values should be determined form observations of light deflection, binary pulsars and cosmological data [39, 40, 41]. Moreover, following Ref. [13], we shall restrict ourselves to spacetimes of small curvatures, for which the conditions

$$|\alpha R| << 1, \qquad |\beta R_{ab}| << 1 \tag{4}$$

hold. Although demanding that the mass scales associated with the linearized equations are real may place additional constraints [42, 43, 44] on α and β, we shall treat them as small but arbitrary.

The tensor F_{ab} satisfies equations

$$\nabla_a \left(\frac{d\mathcal{L}\left(F\right)}{dF} F^{ab}\right) = 0, \tag{5}$$

$$\nabla_a {}^*F^{ab} = 0, \tag{6}$$

and the asterix denotes the Hodge duality. The stress-energy tensor defined as

$$T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} S_m \tag{7}$$

is given therefore by

$$T_a^b = \frac{1}{4\pi} \left(\frac{d\mathcal{L}\left(F\right)}{dF} F_{ca} F^{cb} - \frac{1}{4} \delta_a^b \mathcal{L}\left(F\right)\right). \tag{8}$$

Differentiating functionally the action S with respect to the metric tensor one has

$$L^{ab} = G^{ab} - \alpha I^{ab} - \beta J^{ab} = 8\pi T^{ab}, \tag{9}$$

where

$$I^{ab} = 2R^{;\,ab} - 2RR^{ab} + \frac{1}{2}g^{ab}\left(R^2 - 4\Box R\right) \tag{10}$$

and

$$J^{ab} = R^{;\,ab} - \Box R^{ab} - 2R_{cd}R^{cbda} + \frac{1}{2}g^{ab}\left(R_{cd}R^{cd} - \Box R\right). \tag{11}$$

Let us consider the spherically symmetric and static configuration described by the line element of the form

$$ds^2 = -e^{2\psi(r)}f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{12}$$

where

$$f(r) = 1 - \frac{2M(r)}{r}. \tag{13}$$

The spherical symmetry places restrictions on the components of F_{ab} tensor and its only nonvanishing components compatible with the assumed symmetry are F_{01} and F_{23}. Simple calculations yield

$$F_{23} = Q\sin\theta \tag{14}$$

and

$$r^2 e^{-2\psi}\frac{d\mathcal{L}\left(F\right)}{dF}F_{10} = Q_e, \tag{15}$$

where Q and Q_e are the integration constants interpreted as the magnetic and electric charge, respectively. In the latter we shall assume that the electric charge vanishes, and, consequently, F is given by

$$F = \frac{2Q^2}{r^4}. \tag{16}$$

The stress-energy tensor (7) calculated for this configuration is

$$T_t^t = T_r^r = -\frac{1}{16\pi}\mathcal{L}\left(F\right) \tag{17}$$

and

$$T_\theta^\theta = T_\phi^\phi = \frac{1}{4\pi}\frac{d\mathcal{L}\left(F\right)}{dF}\frac{Q^2}{r^4} - \frac{1}{16\pi}\mathcal{L}\left(F\right). \tag{18}$$

To simplify calculations and to keep control of the order of terms in complicated series expansions we shall introduce a dimensionless parameter ε substituting $\alpha \to \varepsilon\alpha$ and $\beta \to \varepsilon\beta$. We shall put $\varepsilon = 1$ at the final stage of calculations. Of functions $M\left(r\right)$ and $\psi\left(r\right)$ we assume that they can be expanded as

$$M\left(r\right) = M_0\left(r\right) + \varepsilon M_1\left(r\right) + \mathcal{O}\left(\varepsilon^2\right) \tag{19}$$

and

$$\psi\left(r\right) = \varepsilon\psi_1\left(r\right) + \mathcal{O}\left(\varepsilon^2\right). \tag{20}$$

Consider the left hand side of Eq. (9) calculated for the line element (12) first. Making use of the above expansions and collecting the terms with the like power one obtains

$$L_t^t = -\frac{2}{r^2}(M_0' + \varepsilon M_1' - \varepsilon S_t^t), \tag{21}$$

where

$$
\begin{aligned}
S_t^t = \beta \Bigg(& \frac{2\,M_0'}{r^2} - \frac{8\,M_0\,M_0'}{r^3} + \frac{2\,M_0'^2}{r^2} - \frac{2\,M_0''}{r} + \frac{5\,M_0\,M_0''}{r^2} - \frac{M_0'\,M_0''}{r} \\
& + \frac{M_0''^2}{2} + M_0^{(3)} - \frac{M_0\,M_0^{(3)}}{r} - M_0'\,M_0^{(3)} + r\,M_0^{(4)} - 2\,M_0\,M_0^{(4)} \Bigg) \\
& - \alpha \Bigg(\frac{24\,M_0\,M_0'}{r^3} - \frac{8\,M_0'}{r^2} - \frac{4\,M_0'^2}{r^2} + \frac{8\,M_0''}{r} - \frac{18\,M_0\,M_0''}{r^2} - M_0''^2 \\
& + \frac{2\,M_0'\,M_0''}{r} - 4\,M_0^{(3)} + \frac{6\,M_0\,M_0^{(3)}}{r} + 2\,M_0'\,M_0^{(3)} - 2\,r\,M_0^{(4)} + 4\,M_0\,M_0^{(4)} \Bigg)
\end{aligned}
\tag{22}
$$

and M_0', M_0'' and $M_0^{(i)}$ for $i \geq 3$ denote first, second and $i-$th derivatives with respect to the radial coordinate. On the other hand, a simple combination of the components of L_a^b tensor

$$L_r^r - L_t^t = 0 \tag{23}$$

can be easily integrated to yield

$$\psi_1(r) \;=\; (2\alpha+\beta)M_0^{(3)} - \frac{4}{r^2}(3\alpha+\beta)M_0' + C_1, \tag{24}$$

where C_1 is the integration constant. It should be noted that contrary to the case of coupled system of the Maxwell equations and quadratic gravity considered in Refs. [11, 12, 13, 15] now we have explicit dependence on the parameter α. This together with the nonlinear character of the source term (that will be specified below) results in substantial complications of the first-order equations.

To develop the model further we shall express solutions of the system of differential equations consisting of the time component of Eqs. (9) and Eq. (23) in terms of the total mass $\mathcal{M}$ as seen by a distant observer

$$\lim_{r\to\infty} M\left(r\right) = \mathcal{M}, \tag{25}$$

whereas for the function $\psi\left(r\right)$ we shall adopt the natural condition

$$\lim_{r\to\infty} \psi\left(r\right) = 0. \tag{26}$$

III. SOLUTIONS

Further considerations require specification of the Lagrangian $\mathcal{L}\left(F\right)$. Following Ayón-Beato, García and Bronnikov let us chose it in the form

$$\mathcal{L}\left(F\right) \,=F\left[1-\tanh^2\left(s\,\sqrt[4]{\frac{Q^2F}{2}}\right)\right], \tag{27}$$

where

$$s = \frac{|Q|}{2b}, \tag{28}$$

and the free parameter b will be adjusted to guarantee regularity at the center.

Before proceeding further we shall briefly discuss the question of regularity, postponing presentation of the technical details for a while. First, observe that the zeroth-order solution coincides with a general ABGB line element that depends on a free parameter b and an integration constant C_2. On the other hand, the first-order solution written in the suitable representation approaches, as we shall see, a constant value, say μ, as $r \to \infty$, and, consequently, by the boundary conditions (25) one has $C_2 = \mathcal{M} - \varepsilon\mu$. The thus obtained line element is generally singular at the center, and the only way to make it regular consists in appropriate choice of the free parameter. The regularity is understood here as the regularity of the line element rather than regularity of the spacetime itself as the latter requires the curvature invariants to be regular. We shall return to this issue later.

Now we present the calculations in a more systematic form. In order to guarantee sufficient generality of our considerations we shall take the parameter b in the form

$$b = b_1 + \varepsilon b_2, \tag{29}$$

with $b_2 \neq 0$. Since we have assumed the expansions of $M(r)$ and $\psi(r)$ in the form given by Eqs. (19) and (20), respectively, we shall rewrite the boundary conditions as

$$\lim_{r\to\infty} M_0(r) = \mathcal{M}, \tag{30}$$

with vanishing $M_1(\infty)$ and $\psi_1(\infty)$.

Inserting Eq. (28) into (27) and makig use of Eq. (16) one has

$$\mathcal{L}(F) = \frac{2Q^2}{r^4}\left(1 - \tanh^2\frac{Q^2}{2br}\right). \tag{31}$$

Subsequently expanding the right hand side of Eqs. (9) with respect to ε and retaining the linear terms only yields

$$8\pi T_t^t = 8\pi T_r^r = -\frac{Q^2}{r^4}\left(1 - \tanh^2\frac{Q^2}{2b_1 r}\right) - \varepsilon\frac{b_2 Q^4}{b_1^2 r^5}\left(\cosh^{-2}\frac{Q^2}{2b_1 r}\tanh\frac{Q^2}{2b_1 r}\right). \tag{32}$$

The zeroth-order equation

$$M_0'(r) = \frac{Q^2}{2r^2}\left(1 - \tanh^2\frac{Q^2}{2b_1 r}\right)$$

can be easily integrated

$$M_0(r) = C_2 - b_1\tanh\frac{Q^2}{2b_1 r}, \tag{33}$$

where C_2 is the integration constant. Making use of the condition (30) gives $C_2 = \mathcal{M}$. On the other hand, demanding of the regularity of the line element as $r \to 0$ yields $b_1 = \mathcal{M}$, and, consequently, $M_0(r)$ reads

$$M_0(r) = \mathcal{M}\left(1 - \tanh\frac{Q^2}{2\mathcal{M}r}\right). \tag{34}$$

The zeroth-order solution has an interesting property, which, as we shall see, is crucial in our subsequent analysis: $M_0(r)$ as well as its derivatives vanish in the limit $r \to 0$. Moreover, it should be emphasized that although the integration constant and the free parameter in the thus constructed solution are equal to the total mass of the system as seen by a distant observer their status is different: the interpretation of the former is an inevitable consequence of the boundary conditions whereas the latter should be postulated.

For small values of $|Q|/\mathcal{M}$ as well as at great distances the ABGB line element resembles that of Reissner-Nordström. It can be easily seen by expanding the function $M_0(r)$

$$g_{tt} = -1 + \frac{2\mathcal{M}}{r} - \frac{Q^2}{r^2} + \frac{Q^6}{12\mathcal{M}^2 r^4} + ..., \tag{35}$$

whereas for $r \to 0$ one has

$$f \sim 1 - \frac{4\mathcal{M}}{r}\exp\left(\frac{-Q^2}{\mathcal{M}r}\right). \tag{36}$$

Noticeable differences appear for the configurations near the extremality limit and in the internal region in the vicinity of the center. Indeed, for the ABGB solution g_{tt} tends to -1 as $r \to 0$ whereas the (00) component of the metric tensor of the Reissner-Nordström solution diverges in that limit as $-r^{-2}$.

From (24) one concludes that the zeroth-order solution, $M_0(r)$, is sufficient to determine the function $\psi_1(r)$. Indeed, substituting (34) into Eq. (24) and making use of the condition (26) one obtains

$$\begin{aligned}\psi_1(r) \;=\; & \left[\frac{\beta Q^2}{r^4} + \frac{(2\alpha+\beta)Q^6}{2\mathcal{M}^2 r^6}\tanh^2\frac{Q^2}{2\mathcal{M}r} - \frac{3(2\alpha+\beta)Q^4}{\mathcal{M}r^5}\tanh\frac{Q^2}{2\mathcal{M}r}\right]\cosh^{-2}\frac{Q^2}{2\mathcal{M}r} \\ & - \frac{(2\alpha+\beta)Q^6}{4\mathcal{M}^2 r^6}\cosh^{-4}\frac{Q^2}{2\mathcal{M}r}.\end{aligned} \tag{37}$$

It could be easily demonstrated that that $\psi_1(r)$ vanishes at $r = 0$ and inspection of Eq. (24) reveals similar behaviour of its derivatives.

The solution of the first-order equation

$$\frac{2}{r^2}(M_1' - S_t^t) = \frac{b_2 Q^4}{b_1^2 r^5}\left(\cosh^{-2}\frac{Q^2}{2b_1 r}\,\tanh\frac{Q^2}{2b_1 r}\right) \tag{38}$$

is more complicated and when combined with the appropriate boundary conditions it could be written as

$$M_1(r) = \int_\infty^r S_t^t(r)\,dr + \frac{b_2 Q^4}{2\mathcal{M}^2}\int_\infty^r \frac{1}{r^3}\left(\cosh^{-2}\frac{Q^2}{2\mathcal{M}r}\,\tanh\frac{Q^2}{2\mathcal{M}r}\right)dr. \tag{39}$$

Let us consider the first integral in the right hand side of (39) first. The result can be expressed in terms of the hyperbolic functions and polylogarithms $\mathrm{Li}_i(s)$. Indeed, utilizing formulas collected in Appendix one can construct the solution which has the general structure

$$\begin{aligned} M_1^{(0)}(r) &= -\int_\infty^r S_t^t(r)\,dr = \frac{1}{\mathcal{M}}\sum_{i=0}^{1}\sum_{j=0}^{2}\left(\alpha\widetilde{f}_{ij}^{(\alpha)} + \beta\widetilde{f}_{ij}^{(\beta)}\right)\tanh^i\frac{Q^2}{2\mathcal{M}r}\,\mathrm{sech}^{2j}\frac{Q^2}{2\mathcal{M}r} \\ &+ \frac{1}{\mathcal{M}}\sum_{i=1}^{6}\left(\alpha\widetilde{h}_i^{(\alpha)} + \beta\widetilde{h}_i^{(\beta)}\right)\mathrm{Li}_i\left(-\exp\left(-\frac{Q^2}{\mathcal{M}r}\right)\right) - \mu. \end{aligned} \tag{40}$$

Here μ is given by

$$\mu = \frac{\pi^2\mathcal{M}^5}{Q^6}\left[\alpha\left(8 - \frac{31}{315}\pi^4\right) + \beta\left(\frac{8}{3} + \frac{7}{45}\pi^2 - \frac{31}{630}\pi^4\right)\right] = \frac{\pi^2\mathcal{M}^5}{Q^6}\sigma, \tag{41}$$

and since the terms in the square brackets will appear frequently in our subsequent analyses we have singled them out and denoted by σ. The functions $\tilde{f}$ and $\tilde{h}$ are simple polynomials of r^{-1}; their actual form will not be displayed here as we shall readily rewrite them in a slightly modified form. It should be noted, however, that a careful term-by-term analysis of Eq. (40) reveals that $M_1^{(0)}(r)$ approaches $-\mu$ as $r \to 0$. Since for $b = b_1$ the functions $M_1(r)$ and $M_1^{(0)}$ coincide it is impossible to obtain a regular solution at the center, and the remedy is to retain b_2 in the calculations.

Converting all hyperbolic functions appearing in the solution to the exponents, introducing a new function ξ defined as

$$\xi(r) = \exp\left(-\frac{Q^2}{\mathcal{M}r}\right) \tag{42}$$

and finally making use of the elementary properties of the function $\mathrm{Li}_0(-\xi)$ (see Eq. (A.11) of the Appendix) one has

$$M_1^{(0)}(r) = \sum_{i=0}^{6}\left(\alpha f_i^{(\alpha)} + \beta f_i^{(\beta)}\right)\mathrm{Li}_i(-\xi) - \mu. \tag{43}$$

The term μ has been singled out for convenience and $f_i^{(\alpha)}$ and $f_i^{(\beta)}$ are given respectively by

$$\begin{aligned} f_0^{(\alpha)} &= -\frac{48\mathcal{M}^3}{Q^2 r^2} + \frac{4\mathcal{M}Q^2(\xi-1)}{r^4(1+\xi)^2} - \frac{16\mathcal{M}^2}{r^3(1+\xi)} - \frac{4Q^6\left(1-4\xi+\xi^2\right)}{\mathcal{M}^2 r^5(1+\xi)^3} \\ &- \frac{4Q^4\left[\mathcal{M}\left(1+86\xi-89\xi^2\right)+25r\left(\xi^2-1\right)\right]}{5\mathcal{M}r^5(1+\xi)^3} + \frac{2Q^6\xi\left(75-425\xi+125\xi^2+\xi^3\right)}{15\mathcal{M}r^6(1+\xi)^4}, \end{aligned} \tag{44}$$

$$f_1^{(\alpha)} = \frac{4Q^4}{5r^5} - \frac{96\mathcal{M}^4}{Q^4 r}, \qquad f_2^{(\alpha)} = \frac{4\mathcal{M}Q^2}{r^4} - \frac{96\mathcal{M}^5}{Q^6}, \tag{45}$$

$$\frac{6r^3\mathcal{M}^3}{Q^6}f_3^{(\alpha)} = \frac{2r^2\mathcal{M}^2}{Q^4}f_4^{(\alpha)} = \frac{r\mathcal{M}}{Q^2}f_5^{(\alpha)} = f_6^{(\alpha)} = \frac{96\mathcal{M}^5}{Q^6} \tag{46}$$

and

$$f_0^{(\beta)} = -\frac{16\mathcal{M}^3}{Q^2r^2} - \frac{16\mathcal{M}^2}{3r^3(1+\xi)} - \frac{2Q^2\left[12r(1+\xi)+\mathcal{M}\left(3-36\xi+\xi^2\right)\right]}{3r^4(1+\xi)^2}$$
$$-\frac{2Q^6\left(1-4\xi+\xi^2\right)}{\mathcal{M}^2r^5(1+\xi)^3} + \frac{Q^6\xi\left(75-425\xi+125\xi^2+\xi^3\right)}{15\mathcal{M}r^6(1+\xi)^4}$$
$$-\frac{2Q^4\left[\mathcal{M}\left(1+96\xi-109\xi^2\right)+30r\left(\xi^2-1\right)\right]}{5\mathcal{M}r^5(1+\xi)^3}, \tag{47}$$

$$f_1^{(\beta)} = \frac{2Q^4}{5r^5} - \frac{8\mathcal{M}^2}{3r^3} - \frac{32\mathcal{M}^4}{Q^4r}, \qquad f_2^{(\beta)} = -\frac{32\mathcal{M}^5}{Q^6} + \frac{2\mathcal{M}Q^2}{r^4} - \frac{8\mathcal{M}^3}{Q^2r^2}, \tag{48}$$

$$f_3^{(\beta)} = \frac{8\mathcal{M}^2}{r^3} - \frac{16\mathcal{M}^4}{Q^4r}, \qquad f_4^{(\beta)} = -\frac{16\mathcal{M}^5}{Q^6} + \frac{24\mathcal{M}^3}{Q^2r^2}, \tag{49}$$

$$\frac{r\mathcal{M}}{Q^2}f_5^{(\beta)} = f_6^{(\beta)} = \frac{48\mathcal{M}^5}{Q^6}. \tag{50}$$

The second quadrature in the right hand side of Eq. (39) is elementary and gives

$$M_1^{(1)} = \frac{b_2Q^4}{2\mathcal{M}^2}\int_\infty^r \frac{1}{r^3}\left(\cosh^{-2}\frac{Q^2}{2\mathcal{M}r}\tanh\frac{Q^2}{2\mathcal{M}r}\right)dr = \frac{b_2Q^2}{2\mathcal{M}r}\cosh^{-2}\frac{Q^2}{2\mathcal{M}r} - b_2\tanh\frac{Q^2}{2\mathcal{M}r} \tag{51}$$

or equivalently

$$M_1^{(1)} = b_2\left[\frac{1-\xi}{\xi} - \frac{2Q^2}{\mathcal{M}r(1+\xi)}\right]\mathrm{Li}_0(-\xi). \tag{52}$$

Now, in order to obtain a regular solution at the center one has to adjust suitably the parameter b_2. To accomplish this let us observe that $M_1^{(1)}(0) = -b_2$. Further, taking the results of the discussion presented below Eq. (41) into account one finds

$$b_2 = -\mu. \tag{53}$$

It could be easily seen that with such a choice of b_2 the function $M_1(r)$ vanishes at the center and as $r\to\infty$ as required.

It should be noted that the representation of the thus obtained line element is by no means unique. For example one can express the result in terms of $\mathrm{Li_n}(-1/\xi)$ rather than $\mathrm{Li_n}(-\xi)$ employing identity [45]

$$\mathrm{Li}_n(-\xi) + (-1)^n\mathrm{Li}_n(-1/\xi) = -\frac{1}{n!}\ln^n\xi + 2\sum_{r=1}^{[n/2]}\frac{\ln^{n-2r}\xi}{(n-2r)!}\mathrm{Li}_{2r}(-1), \tag{54}$$

where $[n/2]$ is the greatest integer contained in $n/2$ and the constants $\mathrm{Li}_{2r}(-1)$ are related to the Bernoulli numbers, B_{2r}, according to the formula

$$\mathrm{Li}_{2r}(-1) = \frac{2^{2r-1}}{(2r)!}B_{2r}\pi^{2r}. \tag{55}$$

Regardless of the chosen representation, after imposing boundary conditions, both solutions are, of course, equivalent.

As the ABGB geometry reduces to the Schwarzschild solution in the limit $Q = 0$, it is the charge, no matter how small, that secures regularity. The vanishing of the charge leads therefore to dramatic changes in the geometry of the black hole interior. On the other hand, the terms calculated to $\mathcal{O}(Q^2)$ for the ABGB spacetime coincide with the

Reissner-Nordström solution. Let us consider a series expansion of the function $M_1^{(1)}(r)$ for small $q = |Q|/\mathcal{M}$. After some algebra one has

$$\begin{aligned} M_1 \;=\; & \left(\frac{2}{x^3}-\frac{3}{x^4}\right)\frac{\beta}{\mathcal{M}}q^2+\frac{6\beta}{5\mathcal{M}}q^4-\left[\left(\frac{3}{x^5}-\frac{11}{2x^6}\right)\alpha+\left(\frac{9}{4x^5}-\frac{4}{x^6}\right)\beta\right]\frac{q^6}{\mathcal{M}} \\ - & \left(\frac{5}{2}\alpha+\frac{13}{7}\beta\right)\frac{q^8}{\mathcal{M}x^7}\;+\;...-b_2\left(\frac{q^6}{12x^3}-\frac{q^{10}}{60x^5}+\frac{17q^{14}}{6720x^7}\right)+..., \end{aligned} \tag{56}$$

where $x = r/\mathcal{M}$. It could be easily checked that for the regular line element the leading term of the expansion is proportional to b_2Q^6. Since the coefficient b_2 is proportional to Q^{-6} the constructed line element does not approach the Schwarzschild solution in the limit $Q \to 0$. Indeed, simple calculations yield

$$ds^2=-\left(1-\frac{2\mathcal{M}}{r}-\frac{2\mathcal{M}^2}{r^4}k\right)dt^2+\left[\left(1-\frac{2\mathcal{M}}{r}\right)^{-1}+\left(1-\frac{2\mathcal{M}}{r}\right)^{-2}\frac{2\mathcal{M}^2}{r^4}k\right]dr^2+r^2d\Omega^2, \tag{57}$$

where k is given by

$$k=\frac{\pi^2}{12}\sigma\approx 5.781\alpha-0.486\beta. \tag{58}$$

Consequently, one has either Schwarzschild asymptotic of a singular line element with $b_2 = 0$ ($k = 0$) or a regular line element with non-Schwarzschild $Q = 0$ limit for $b_2 = -\mu$. It is possible, of course, to accept other values of the parameter b_2 but it seems that they are of lesser importance.

The approximate location of the event horizon of the line element (57) lies near its Schwarzschild value and is approximately given by

$$r_+\approx 2\mathcal{M}\left(1+\frac{k}{8\mathcal{M}^2}\right). \tag{59}$$

As is well-known, the Hawking temperature, T_H, can be easily calculated from the metric tensor without referring to the field equations. The standard by now method of obtaining T_H relies on the Wick rotation. The Euclidean line element has no conical singularity provided the time coordinate is periodic with a period P given by

$$P=4\pi\lim_{r\to r_+}\left(g_{tt}g_{rr}\right)^{1/2}\left(\frac{d}{dr}g_{tt}\right)^{-1}. \tag{60}$$

Its reciprocal is identified with the black hole temperature, which, in the case in hand, reads

$$T_H\;=\;\frac{1}{8\pi\mathcal{M}}\left(1+\frac{k}{4\mathcal{M}^2}\right). \tag{61}$$

Note that for α and β satisfying

$$\alpha=\frac{1680+98\pi^2-31\pi^4}{2(31\pi^4-2520)}\beta, \tag{62}$$

both (57) and (61) reduce to their Schwarzschild counterparts.

Finally, we shall investigate the important issue of regularity. However, before we proceed further a few words of comments are in order. It should be emphasized that we have constructed a linearized solution to the coupled system of equations of quadratic gravity and nonlinear electrodynamics only. Consequently, when calculating the curvature invariants we are to restrict ourselves to $O(\varepsilon)$ terms. As it has been stressed earlier, the regularity of the line element does not necessarily entail the regularity of the underlying geometry: the latter requires various curvature invariants to be regular in the interesting region. First, let us concentrate on the Kretschmann scalar, K. It could be easily shown that for the constructed line element it consists of terms involving products of the functions $M_0(r)$, $M_1(r)$, $\psi_1(r)$ and their derivatives. Indeed, after some algebra one obtains $K = K_0 + \varepsilon\Delta K$, where K_0 is the Kretschmann scalar of the ABGB spacetime

$$K_0=\frac{48M_0^2}{r^6}+\frac{16M_0''M_0}{r^4}+\frac{32M_0'^2}{r^4}-\frac{64M_0'M_0}{r^5}+\frac{4M_0''^2}{r^2}-\frac{16M_0''M_0'}{r^3} \tag{63}$$

and ΔK is given by

$$\begin{aligned}\Delta K \;=\; & \left(\frac{128 M_0 M_0'}{r^4} + \frac{16 M_0}{r^4} - \frac{80 M_0^2}{r^5} - \frac{16 M_0'}{r^3} - \frac{24 M_0 M_0''}{r^3} + \frac{24 M_0' M_0''}{r^2} - \frac{48 {M_0'}^2}{r^3}\right)\psi_1' \\ & + \left(-\frac{8 M_0''}{r} + \frac{16 M_0 M_0''}{r^2} + \frac{32 M_0^2}{r^4} + \frac{16 M_0'}{r^2} - \frac{32 M_0 M_0'}{r^3} - \frac{16 M_0}{r^3}\right)\psi_1'' \\ & + \left(-\frac{16 M_0''}{r^4} - \frac{96 M_0}{r^6} + \frac{64 M_0'}{r^5}\right) M_1 + \left(\frac{16 M_0'}{r^3} - \frac{16 M_0}{r^4} - \frac{8 M_0''}{r^2}\right) M_1'' \\ & + \left(\frac{16 M_0''}{r^3} - \frac{64 M_0'}{r^4} + \frac{64 M_0}{r^5}\right) M_1'. \end{aligned} \tag{64}$$

The regularity of K_0 has been demonstrated in Ref [31]; in order to establish regularity of ΔK one has to establish regularity of its building blocks. To analyse behaviour of the $k-$th derivative of M_1 we shall employ the first order equation. From Eq. (22) one clearly sees that all derivatives of S_t^t vanish as $r \to 0$. It is simply because $k-$th derivative of M_0 has the asymptotic form

$$\frac{d^k M_0}{dr^k} \sim \sum_{i=k+1}^{2k} \frac{c_i}{r^i} \exp\left(-\frac{Q^2}{\mathcal{M} r}\right), \tag{65}$$

as $r \to 0$, where the coefficients c_i depend on $\mathcal{M}$ and Q. It follows then that k-th derivative behaves as $r^{-2k}\exp(-Q^2/\mathcal{M}r)$ near the center. One encounters a similar behaviour in the second term in the right hand side of Eq. (39). Pulling this all together one concludes that the Kretschmann scalar is indeed regular at the center. Moreover, one can easily establish the regularity of R^2 and $R_{ab}R^{ab}$. Similar arguments can be used in demonstration that higher invariants constructed from the curvature are also regular.

IV. GEOMETRY

Generally speaking a black hole belongs to one of the two distinct classes: it may be either extremal (when horizon has at least twofold degeneracy) or nonextremal. In the former case the family of ultraextremal black holes, i. e. configurations with triple (or even higher) degeneracy can be singled out [46, 47]. Since we have restricted ourselves to the case of vanishing cosmological constant, we expect that the ABGB black hole possesses at most two horizons. (Generalization of the ABGB solution to $\Lambda \neq 0$ case is straightforward.) Indeed, depending on $q = |Q|/\mathcal{M}$ the ABGB spacetime has two, one, or has no horizons at all. Simple calculations indicate that for $q < q_c$, where q_c is a critical value of q, it has both the inner and outer horizon, whereas for $q > q_c$ the metric potential $g_{tt}(r)$ has no real roots for $r \geq 0$. For $q = q_c$ the event and inner horizons merge and the configuration becomes extremal. It should be noted that the extremal ABGB geometry is distinct from, say, the geometry of the extremal Reissner-Nordström solution. Indeed, because of the regularity of the former, the configurations with $q > q_c$ are not forbidden by a cosmic censor.

A. Nonextreme black holes

In order to determine the location of the inner and outer horizon we shall analyse equation $g_{tt}(r) = 0$ and construct the iterative solution restricting ourselves to the terms linear in ε. As the solution can be written as

$$r_\pm = r_0^{(\pm)} + \varepsilon r_1^{(\pm)}, \tag{66}$$

one has to solve the simple system of two equations. First of them

$$1 - \frac{2 M_0\left(r_0^{(\pm)}\right)}{r_0^{(\pm)}} = 0 \tag{67}$$

admits solutions expressible in terms of the Lambert special functions. Indeed, it has been demonstrated in Ref. [32] that $x_0^{(\pm)} = r_0^{(\pm)}/\mathcal{M}$ are given by

$$x_0^{(\pm)} = -\frac{4q^2}{4W_\pm\left(-\frac{q^2}{4}\exp\left(\frac{q^2}{4}\right)\right) - q^2}, \tag{68}$$

where $W_+(s) \equiv W(0,s)$ and $W_-(s) \equiv W(-1,s)$ are the only real branches of the Lambert functions [48]. On the other hand, r_1 satisfies algebraic equation which can be easily solved. After some manipulations one has

$$r_1^{(\pm)} = \frac{2M_1\left(r_0^{(\pm)}\right)}{1-2M_0'\left(r_0^{(\pm)}\right)}. \tag{69}$$

Now, inserting the zeroth-order solutions (68) into Eq. (69) one can easily determine $r_1^{(\pm)}$, and the result can be further simplified with the substitution

$$x_0 = \frac{4\xi}{\xi+1}, \tag{70}$$

where ξ (defined as in Eq. (42)) is calculated at $x = x_0$. Specifically, denominator in the right hand side of Eq. (69) reduces to the simple expression:

$$1-2M_0'\left(r_0^{(\pm)}\right) = \frac{4\xi-q^2}{4\xi}. \tag{71}$$

To avoid unnecessary proliferation of long formulas we shall not present the results for $r_1^{(\pm)}$ here as they could be readily obtained by a direct substitution of the equation (68) and the expression describing M_1 at $r_0^{(\pm)}$ into Eq. (69).

Although the complexity of the expression describing r_+ makes its direct analytical application rather intricate, one can obtain interesting and important information analyzing limiting cases. Below we derive expansion valid for small q. The extremality limit will be analyzed in the next subsection. Expanding r_+ and collecting the terms with like powers of q, after massive simplifications, one has

$$\begin{aligned} r_+ &= 2\mathcal{M} - \frac{q^2}{2}\mathcal{M} - \frac{q^4}{8}\mathcal{M} - \frac{5q^6}{96}\mathcal{M} + ... + \frac{1}{4\mathcal{M}}k + \left(\frac{1}{4\mathcal{M}}k + \frac{\beta}{8\mathcal{M}}\right)q^2 \\ &+ \left(\frac{71}{320\mathcal{M}}k + \frac{17}{160\mathcal{M}}\beta\right)q^4 + \left(\frac{123}{640\mathcal{M}}k + \frac{47}{640\mathcal{M}}\beta - \frac{1}{64\mathcal{M}}\alpha\right)q^6 + ... \end{aligned} \tag{72}$$

Analogous expression for the event horizon of the Reissner-Nordström solution in quadratic gravity reads

$$\begin{aligned} r_+ &= 2\mathcal{M} - \frac{1}{2}\mathcal{M}q^2 - \frac{1}{8}\mathcal{M}q^4 - \frac{1}{16}\mathcal{M}q^6 + ... \\ &+ \left(\frac{1}{8\mathcal{M}}q^2 + \frac{17}{160\mathcal{M}}q^4 + \frac{57}{640\mathcal{M}}q^6 + ...\right)\beta. \end{aligned} \tag{73}$$

Inspection of (72) and (73) shows that for $b_2 = 0$ (or, equivalently, $k = 0$) both expansion coincide up to q^4 terms.

One of the most important characteristics of the black hole is its temperature. To investigate how it is modified by the quadratic terms, we employ a general expression (60), which, in the present context, can be rewritten in the form

$$T_H = \frac{1}{4\pi}\lim_{r\to r_+} e^{-\varepsilon\psi_1}\left(\frac{1}{r} - \frac{2\varepsilon}{r}S_t^t + 8\pi r T_t^t\right). \tag{74}$$

Further, expanding T_H, in the auxiliary parameter, collecting the terms with like powers of ε and linearizing the thus obtained result one has

$$T_H = T_0 + \varepsilon\Delta T, \tag{75}$$

where T_0 coincides with the temperature of the nonextremal ABGB black hole

$$T_0 = \frac{1}{4\pi}\left[\frac{1}{r_0} - \frac{Q^2}{\mathcal{M}r_0^2}\left(1-\frac{r_0}{4\mathcal{M}}\right)\right] \tag{76}$$

and ΔT is given by

$$\Delta T = r_1\left(2r_0\frac{dT_t^t}{dr}_{|r_0} - \frac{1}{2\pi r_0^2} + \frac{T_0}{r_0}\right) - \frac{1}{2\pi r_0}S_t^t + T_0\psi_1(r_0) + 2r_0T_t^t. \tag{77}$$

Here $r_0 = r_0^{(+)}$, whereas $T^t_{(0)t}$ and $T^t_{(1)t}$ are given by the first and the second term in the right hand side of Eq. (32), respectively.

Of course, the general expression for temperature is also too complicated to be displayed here. We shall, therefore, calculate its limiting behaviour for small q precisely in the same manner as it has been done with the location of the event horizon r_+. As the temperature of the extremal configuration is zero, this condition can be used in establishing relation between Q and $\mathcal{M}$. This will be done in the next subsection. The expansion of the temperature of the nonextremal black hole reads

$$\begin{aligned} T_H &= \frac{1}{8\pi\mathcal{M}} - \frac{q^4}{128\pi\mathcal{M}} - \frac{5q^6}{768\pi\mathcal{M}} + ... \\ &+ \frac{k}{32\pi\mathcal{M}^3} + \frac{q^2}{64\pi\mathcal{M}^3}(3k-\beta) + \frac{q^4}{80\pi\mathcal{M}^3}\left(\frac{33}{8}k - \beta\right) + ... \end{aligned} \tag{78}$$

whereas for the Reissne-Norström geometry in quadratic gravity one has

$$T_H = \frac{1}{8\pi\mathcal{M}} - \frac{q^4}{128\pi\mathcal{M}} - \frac{q^6}{128\pi\mathcal{M}} + \ ... + \left(\frac{q^4}{320\mathcal{M}} + \frac{3q^6}{512\mathcal{M}} + ...\right)\beta. \tag{79}$$

B. Extreme black holes

In this section we shall investigate the important issue of extremal black holes. When r_+ and r_- of the classical ABGB black hole merge into the one degenerate horizon one has the extremal configuration characterized by vanishing surface gravity (Hawking temperature). The geometry of the vicinity of the degenerate horizon belongs to the $\mathrm{AdS}_2 \times \mathrm{S}^2$ class with different modules of curvatures of subspaces [49]. One of the peculiarities of the general extremal solution is infinite proper distance between two points, one of which resides on the event horizon. The degeneracy means that in the vicinity of the horizon the leading behaviour of the metric potential g_{tt} is of the form $g_{tt} \sim (r - r_+)^2$, or equivalently

$$g_{tt}(r_+) = g'_{tt}(r_+) = 0. \tag{80}$$

Combining these equations one obtains simple algebraic equation that could be easily solved

$$x_+ = \frac{4q_c^2}{4 - q_c^2}. \tag{81}$$

It can be demonstrated that the parameters q and x_x of the classical extremal ABGB black hole can be expressed in terms of the principal branch of the Lambert function evaluated at e^{-1}. Employing definition of W_+ one has

$$q_c = 2\sqrt{W_+(e^{-1})} \approx 1.0554, \tag{82}$$

whereas the location of the degenerate horizon is given by

$$x_c = \frac{4W_+(e^{-1})}{1 + W_+(e^{-1})} \approx 0.8712. \tag{83}$$

Here and below x (with or without sub or superscripts) denotes appropriate radial coordinate divided by $\mathcal{M}$.

Now, we shall examine modifications caused by quadratic gravity. Lengthy but routine calculations show that the extremal canfiguration is possible for

$$q^2 = q_c^2 + \delta, \tag{84}$$

where the correction δ is given by

$$\delta = \frac{1}{\mathcal{M}^2}\sum_{i=0}^{6}\left(\alpha h_i^{(\alpha)} + \beta h_i^{(\beta)}\right)\mathrm{Li}_i(w) \ + \ \frac{2(3w-1)}{\mathcal{M}}b_2 \tag{85}$$

with

$$h_0^{(\alpha)} = -\frac{(1+w)^3}{240w^3}\left(516 - 69w + 11w^2 - 5w^3 - w^4\right) - \frac{(1+w)^2\pi^2}{10080w^4}\left(31\pi^2 - 2520\right), \tag{86}$$

$$h_1^{(\alpha)} = -\frac{(1+w)^2}{40w^3}\left(119 - 4w - 6w^2 - 4w^3 - w^4\right), \tag{87}$$

$$h_2^{(\alpha)} = -\frac{1}{8w^3}(1+w)\left(23 - 4w - 6w^2 - 4w^3 - w^4\right), \tag{88}$$

$$\frac{2}{(1+w)^3}h_3^{(\alpha)} = \frac{2}{(1+w)^2}h_4^{(\alpha)} = \frac{1}{(1+w)}h_5^{(\alpha)} = h_6^{(\alpha)} = \frac{3(1+w)}{w^3}, \tag{89}$$

$$h_0^{(\beta)} = -\frac{(1+w)^3}{480w^3}\left(356 + 21w + 21w^2 - 5w^3 - w^4\right) + \frac{(1+w)^2\pi^2}{20160w^4}\left(1680 + 98\pi^2 - 31\pi^4\right), \tag{90}$$

$$h_1^{(\beta)} = -\frac{(1+w)^2}{240w^3}\left(257 + 28w + 2w^2 - 12w^3 - 3w^4\right), \tag{91}$$

$$h_2^{(\beta)} = -\frac{1}{16w^3}(1+w)\left(19 + 4w + 2w^2 - 4w^3 - w^4\right), \tag{92}$$

$$h_3^{(\beta)} = -\frac{1}{4w^3}(1+w)^2\left(1 - 2w - w^2\right), \qquad h_4^{(\beta)} = \frac{1}{4w^3}(1+w)\left(1 + 6w + 3w^2\right), \tag{93}$$

$$\frac{1}{1+w}h_5^{(\beta)} = h_6^{(\beta)} = \frac{3(1+w)}{2w^3}, \tag{94}$$

and

$$w = W_+\left(e^{-1}\right). \tag{95}$$

The term containing $b_2(=-\mu)$, i. e., the last term in right hand side of Eq. (85) has been singled out for convenience. For the extreme black hole μ reads

$$\mu = \frac{\pi^2}{64\mathcal{M}w^3}\sigma, \tag{96}$$

where σ is defined through the relation (41). The location of the degenerate horizon is given by

$$x = x_c + \Delta, \tag{97}$$

where Δ can be compactly expressed in the form

$$\Delta = \frac{\delta w}{(1+w)^2} + \frac{2\alpha+\beta}{16w\mathcal{M}^2}(w+3)(w-1) - \frac{4w(w-1)}{\mathcal{M}(1+w)^2}b_2, \tag{98}$$

with δ given by Eq. (85). As both δ and Δ depend on the particular value of the Lambert function at e^{-1} one can easily calculate their numerical values. Indeed, simple calculations yield

$$\delta = -\frac{0.81451}{\mathcal{M}^2}\beta - \frac{3.67845}{\mathcal{M}^2}\alpha \tag{99}$$

and

$$\Delta = \frac{1.40646}{\mathcal{M}^2}\beta + \frac{3.88206}{\mathcal{M}^2}\alpha, \tag{100}$$

for $b_2 = -\mu$, whereas with $b_2 = 0$ one has

$$\delta = -\frac{0.57553}{\mathcal{M}^2}\beta - \frac{0.05121}{\mathcal{M}^2}\alpha \tag{101}$$

and

$$\Delta = -\frac{0.43288}{\mathcal{M}^2}\beta - \frac{1.05314}{\mathcal{M}^2}\alpha. \tag{102}$$

Note that for particular choices of the coupling parameters it is possible to reduce either x_+ or q_c to its general relativistic values. Obtained corrections can be contrasted with the analogous results calculated for the extreme Reissner-Nordström black hole in quadratic gravity:

$$q_c^2 = 1 + \frac{2}{5\mathcal{M}^2}\beta \tag{103}$$

and

$$x_+ = 1 + \frac{1}{5\mathcal{M}^2}\beta. \tag{104}$$

V. CONCLUSION AND SUMMARY

In this paper we have constructed perturbative solutions describing spherically symmetric and static black holes to the coupled equations of fourth-order gravity with the source term given by the stress- energy tensor of the nonlinear electrodynamics. The Lagrangian of the nonlinear field is a natural generalization of the Bronnikov's Lagrangian. Because of technical complexities we have restricted ourselves to the first-order corrections, i. e. the terms proportional to ε. The obtained line element is parametrized by two integration constants and free parameters. Integration constants are related to the (magnetic) charge and a total mass of the system as seen by a distant observer whereas the free parameters are adjusted to make the resulting solution regular at the center. The regularity heavily relies on the form of the zeroth-order solution which coincides with the solution of the ABGB-type and the special properties of the polylogarithms Li_i. The metric potentials thus computed enabled construction of the basic characteristics of the geometry and its asymptotics: location of the inner and outer horizon, Hawking temperature and the relation between Q and $\mathcal{M}$ for the extremal configurations.

If one is interested in the solution possessing Schwarzschild asymptotics as $Q \to 0$ or intend to study the external region without any relations to the issue of regularity it suffices to put $b_2 = 0$ throughout the paper. We intentionally presented basic formulas in a form allowing for a different choices of the parameter b_2. It should be stressed that the only choice leading to regularity of the solution at the center is $b_2 = -\mu$.

The calculations and results presented in this paper suggest some generalizations and obvious extensions. First, one may attempt to go beyond first order calculations in order to establish full regularity of the solution. Of course it would be interesting and desirable, with all conceptual limitations of the method, to demonstrate it in a nonperturbative way. However, the technical complexities may be a real obstacle in this regard. Further, it is possible to analyse behaviour of the test quantum fields in the ABGB background with the special emphasis put on the back reaction on the metric. Indeed, since the general expression describing the stress-energy tensor of quantized massive scalar, spinor and vector fields in the large mass limit is known [32, 50] it is possible to investigate the resultant quantum-corrected geometry even in the vicinity of the center of the black hole. Recently such a programme has been carried out in the case of the quantum-corrected Reissner-Nordström black holes (see Refs. [51, 52] and the references cited therein). Finally, it should be noted that another important choice of the boundary conditions

$$M_i(r_+) = \begin{cases} \frac{r_+}{2} & \text{if } i = 0, \\ 0 & \text{if } i \geq 1 \end{cases} \tag{105}$$

which is, in turn, related to the horizon defined mass of the black hole is not considered in this paper. Here we remark only that parametrizing solution by the exact location of the event horizon and the charge is quite natural and certainly deserves further study. A lesson that follows from investigations of the corrected Reissner-Nordström solution is that some relations, as for example these for the extremal configurations, are simpler and physically more transparent. We intend to return to these problems elsewhere.

APPENDIX

In this appendix we sketch the method of evaluating the indefinite integrals appearing in the solution of the equation (38). After substitution of auxiliary variable $u = x^{-1}$ we have to find an expression for the integral of the form:

$$\int u^p (\tanh u)^s du, \tag{A.1}$$

with natural p, s. We show that when $s > 1$ the above integral can be reduced to the integral with $s = 1$ and then we will express the latter by a sum containing polylogarithm functions. At the beginning we use the formulas (1.4.22.3) and (1.4.22.4) from the very extensive tables of integrals [53]:

$$\int u^p (\tanh u)^{2n} du = \frac{1}{p+1} u^{p+1} + \sum_{k=0}^{n-1} (-1)^{n+k} \binom{n}{k} \int \frac{u^p}{(\cosh u)^{2n-2k}} du, \tag{A.2}$$

$$\begin{aligned} \int u^p (\tanh u)^{2n+1} du \;&=\; \int u^p \tanh u \, du + \sum_{k=0}^{n-1} \frac{(-1)^{n+k}}{2n-2k} \binom{n}{k} \left[-\frac{u^p}{(\cosh u)^{2n-2k}} \right. \\ &\left. + p \int \frac{u^{p-1}}{(\cosh u)^{2n-2k}} du \right]. \end{aligned} \tag{A.3}$$

Evaluation of integrals containing even powers $2n - 2k > 2$ of sech function can be further reduced to the case with the square of sech in the integrand by repeated use of the formula (1.4.24.1) of [53]:

$$\begin{aligned} \int \frac{u^p}{(\cosh u)^q} du \;&=\; \frac{p u^{p-1}}{(q-1)(q-2)(\cosh u)^{q-2}} + \frac{u^p}{(q-1)\sinh u (\cosh u)^{q-1}} \\ &- \frac{p(p-1)}{(q-1)(q-2)} \int \frac{u^{p-2}}{(\cosh u)^{q-2}} du + \frac{q-2}{q-1} \int \frac{u^p}{(\cosh u)^{q-2}} du. \end{aligned} \tag{A.4}$$

As a simple integration by parts yields:

$$\int \frac{u^p}{(\cosh u)^2} du = u^p \tanh u - p \int u^{p-1} \tanh u \, du, \tag{A.5}$$

we clearly see that to accomplish our calculation we have to find the integral:

$$\int u^p \tanh u \, du \tag{A.6}$$

which requires special treatment. It will be proven below that it is expressible in terms of the polylogarithm functions $\mathrm{Li}_n(z), n = 1, 2, ..., p+1$, with a properly chosen argument z. There is considerable literature on analytical and numerical properties of these functions [45, 54] being a generalization of Euler's dilogarithm. In the unit circle polylogarithm of integral order $m > 1$ can be defined by

$$\mathrm{Li}_m(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m}. \tag{A.7}$$

From this series we easily derive the important recurrence relation:

$$\frac{d}{dz} \mathrm{Li}_n(z) = \frac{1}{z} \mathrm{Li}_{n-1}(z) \tag{A.8}$$

which will also be used in the integral form:

$$\mathrm{Li}_n(z) = \int \frac{\mathrm{Li}_{n-1}(z)}{z} dz + C. \tag{A.9}$$

Using this property and the integral representation of dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t} \, dt \tag{A.10}$$

one can extend the definition of polylogarithms to low orders:

$$\mathrm{Li}_1(z) = -\ln(1-z),\ \ \mathrm{Li}_0(z) = \frac{z}{1-z}. \tag{A.11}$$

Starting from the recurrence relation in the integral form we repeatedly integrate by parts using the expression for the derivative of a polylogarithm (A.8) at every step:

$$\begin{aligned}
\mathrm{Li}_n(z) &= C + \int (\ln z)' \,\mathrm{Li}_{n-1}(z) dz = ... = \\
&= C + \ln z \,\mathrm{Li}_{n-1}(z) - \frac{1}{2}(\ln z)^2 \,\mathrm{Li}_{n-2}(z) + \frac{1}{2\cdot 3}(\ln z)^3 \,\mathrm{Li}_{n-3}(z) + ... \\
&\quad + \frac{(-1)^{n-1}}{(n-2)!}(\ln z)^{n-2}\,\mathrm{Li}_2(z) + \frac{(-1)^{n-1}}{(n-1)!}(\ln z)^{n-1}\ \ln(1-z) + \frac{(-1)^{n-1}}{(n-1)!}\int \frac{(\ln z)^{n-1}}{1-z} dz,
\end{aligned} \tag{A.12}$$

where in the final step we have taken into account the form of $\mathrm{Li}_1(z)$. If we now set $z = -e^{-2u}$ we recover the required integral in the last term:

$$(-1)^{n-1}\int \frac{(\ln z)^{n-1}}{1-z} dz = 2^n \int u^{n-1} \frac{e^{-u}}{e^u + e^{-u}} du = 2^{n-1}\left[\frac{u^n}{n} - \int u^{n-1} \tanh u \, du\right]. \tag{A.13}$$

After rearrangement of the sum in the identity (A.12) and change $n-1 \to p$ we get the expression:

$$\int u^p \tanh u \, du \ = \ C + \frac{u^{p+1}}{p+1} + u^p \ln(1+e^{-2u}) - \sum_{k=1}^{p} \frac{p!}{2^k (p-k)!}\, u^{p-k}\,\mathrm{Li}_{k+1}(-e^{-2u}). \tag{A.14}$$

The hyperbolic functions and monomials in u variable arising in our calculation can also be cast in the unified way when the special forms of low-order polylogarithms (A.11) are used.
The case studied here falls into a wide range of expressions which can be integrated with the help of polylogarithms. Some occurrences of these functions in physical problems are mentioned in the Levin's book [45]. It is remarkable that they arise in Feynman diagrams integrals, in particular in computation of quantum electrodynamics corrections to the electron gyromagnetic ratio [55].

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[1] G. T. Horowitz and R. C. Myers, Gen. Rel. Grav. **27**, 915 (1995), gr-qc/9503062.
[2] R. Utiyama and B. S. DeWitt, J. Math. Phys. **3**, 608 (1962).
[3] K. S. Stelle, Phys. Rev. D (3) **16**, 953 (1977).
[4] K. S. Stelle, General Relativity and Gravitation **9**, 353 (1978).
[5] H. Weyl, Ann. der Phys. **59**, 101 (1919).
[6] H. Weyl, Phys. Zeits. **22**, 473 (1921).
[7] W. Pauli, Phys. Zeits. **20**, 457 (1919).
[8] A. S. Eddington, *The mathematical theory of relativity* (Cambridge University Press, Cambridge, 1924).
[9] R. Schimming and H.-J. Schmidt, NTM Schriftenr. Gesch. Naturw. Tech. Med. **27**, 41 (1990), gr-qc/0412038.
[10] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1984), ISBN 0-521-27858-9.
[11] A. Economou and C. O. Lousto, Phys. Rev. **D49**, 5278 (1994), gr-qc/9310021.
[12] M. Campanelli, C. O. Lousto, and J. Audretsch, Phys. Rev. **D49**, 5188 (1994), gr-qc/9401013.
[13] M. Campanelli, C. O. Lousto, and J. Audretsch, Phys. Rev. **D51**, 6810 (1995), gr-qc/9412001.
[14] B. Holdom, Phys. Rev. **D66**, 084010 (2002), hep-th/0206219.
[15] J. Matyjasek and D. Tryniecki, Phys. Rev. **D69**, 124016 (2004), gr-qc/0402098.
[16] K. A. Bronnikov, V. N. Melnikov, G. N. Shikin, and K. P. Staniukowicz, Annals Phys. **118**, 84 (1979).
[17] K. A. Bronnikov, Phys. Rev. Lett. **85**, 4641 (2000).
[18] K. A. Bronnikov, Phys. Rev. **D63**, 044005 (2001), gr-qc/0006014.
[19] A. Burinskii and S. R. Hildebrandt, Phys. Rev. **D65**, 104017 (2002), hep-th/0202066.

[20] A. D. Sakharov, Sov. Phys. JETP **22**, 21 (1966).
[21] E. B. Gliner, Sov. Phys. JETP **22**, 378 (1966).
[22] J. M. Bardeen, in *GR 5 Proceedings* (Tbilisi, 1968).
[23] V. P. Frolov, M. A. Markov, and V. F. Mukhanov, Phys. Lett. **B216**, 272 (1989).
[24] V. P. Frolov, M. A. Markov, and V. F. Mukhanov, Phys. Rev. **D41**, 383 (1990).
[25] I. Dymnikova, Gen. Rel. Grav. **24**, 235 (1992).
[26] I. Dymnikova, Int. J. Mod. Phys. **D12**, 1015 (2003), gr-qc/0304110.
[27] A. Borde, Phys. Rev. **D55**, 7615 (1997), gr-qc/9612057.
[28] M. Mars, M. M. Martín-Prats, and J. M. M. Senovilla, Classical Quantum Gravity **13**, L51 (1996).
[29] E. Ayon-Beato and A. Garcia, Phys. Lett. **B493**, 149 (2000), gr-qc/0009077.
[30] E. Ayon-Beato and A. Garcia, Gen. Rel. Grav. **37**, 635 (2005), hep-th/0403229.
[31] E. Ayon-Beato and A. Garcia, Phys. Lett. **B464**, 25 (1999), hep-th/9911174.
[32] J. Matyjasek, Phys. Rev. **D63**, 084004 (2001), gr-qc/0010097.
[33] W. Berej and J. Matyjasek, Phys. Rev. **D66**, 024022 (2002), gr-qc/0204031.
[34] J. Z. Simon, Phys. Rev. **D41**, 3720 (1990).
[35] J. Z. Simon, Phys. Rev. **D43**, 3308 (1991).
[36] L. Parker and J. Z. Simon, Phys. Rev. **D47**, 1339 (1993), gr-qc/9211002.
[37] M. Visser, Phys. Rev. **D48**, 583 (1993), hep-th/9303029.
[38] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., San Francisco, Calif., 1973).
[39] E. C. de Rey Neto, O. D. Aguiar, and J. C. N. de Araujo, Class. Quant. Grav. **20**, 2025 (2003), gr-qc/0304091.
[40] A. Accioly and H. Blas, Phys. Rev. **D64**, 067701 (2001), gr-qc/0107003.
[41] M. B. Mijic, M. S. Morris, and W.-M. Suen, Phys. Rev. **D34**, 2934 (1986).
[42] N. H. Barth and S. M. Christensen, Phys. Rev. D **28**, 1876 (1983).
[43] B. Whitt, Phys. Lett. **B145**, 176 (1984).
[44] J. Audretsch, A. Economou, and C. O. Lousto, Phys. Rev. **D47**, 3303 (1993), gr-qc/9301024.
[45] L. Levin, *Polylogarithms and Associated Functions* (North Holland, New York, 1981).
[46] M. Visser, Phys. Rev. **D48**, 5697 (1993), hep-th/9307194.
[47] J. Matyjasek and O. B. Zaslavskii, Phys. Rev. **D71**, 087501 (2005), gr-qc/0502115.
[48] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, Adv. Comput. Math. **5**, 329 (1996).
[49] J. Matyjasek, Phys. Rev. **D70**, 047504 (2004), gr-qc/0403109.
[50] J. Matyjasek, Phys. Rev. **D61**, 124019 (2000), gr-qc/9912020.
[51] J. Matyjasek and O. B. Zaslavskii, Phys. Rev. **D64**, 104018 (2001), gr-qc/0102109.
[52] W. Berej and J. Matyjasek, Acta Phys. Polon. **B34**, 3957 (2003).
[53] A. P. Prudnikov, Y. A. Brychkov, and O. J. Marinov, *Integrals and Series, vol. 1 Elementary Functions* (Fizmatlits, Moscow, 2003).
[54] L. Levin, *Structural Properties of Polylogarithms* (American Mathematical Society, Providence, 1991).
[55] S. Laporta and E. Remiddi, Phys. Lett. **B379**, 283 (1996), hep-ph/9602417.